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Nonradial Solutions to a Linear Elliptic Equation with Symmetric Weight

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1 Introduction

In this paper we consider the eigenvalue problem

$$\Delta u + \lambda K(|x|)u = 0 \quad \text{in } \mathbf{R}^n \tag{1.1}$$

in the space

$$\mathcal{D} = \{u \,|\, u \text{ is measurable, } \int_{\mathbf{R}^n} K(|x|)u^2\, dx < \infty\}, \tag{1.2}$$

where $n \geq 3$ and $\lambda \in \mathbf{R}$ is a parameter. We assume that the weight function $K(r)$, $r = |x|$, satisfies

$$\text{(K)} \qquad \begin{cases} K(r) > 0 \quad \text{on } (0,\infty); \\ K(r) \in C((0,\infty)); \\ rK(r) \in L^1(0,\infty). \end{cases}$$

In this paper, we are concerned with solutions of (1.1) in $\mathcal{D}$ and obtain a complete orthogonal basis in $\mathcal{D}$. This problem is analogous to that of the vibration of a disklike membrane or of the linear Schrödinger equation, which is a classical one and well-investigated. All the eigenfunctions can be expressed as a product of the Bessel functions and functions of the argument θ (see e.g., §30 of Farlow [3]). Moreover, they form a complete orthonormal basis. In this context, our problem is the eigenvalue problem of the "oscillation of the whole space". Compared to the case of bounded domains, there seems to be very few results concerning the eigenvalue problem on unbounded domains.

Since the weight K is radially symmetric, (1.1) can have radial solutions which is obtained as a solution of the initial value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + \lambda K(r) u = 0, \quad r > 0, \\ u(0) = 1. \end{cases} \tag{1.3}$$

We note that (1.3) has a unique global solution for any $\lambda > 0$ under the assumption (K). Under a stronger condition than (K), Naito [9] showed by the shooting method that there exists a first "eigenvalue" $\lambda_0 > 0$ for which (1.3) has a positive solution satisfying $\lim_{r\to\infty} r^{n-2}|u| < \infty$. Later, Edelson and Rumbos [2] showed that the first eigenvalue is simple in the class of radial solutions. Recently, Kabeya [6] obtained the following result concerning radial eigenfunctions.

Theorem A (Kabeya [6], Theorem 1) *Suppose that (K) holds and $n > 2$. Then there exists a unique monotone increasing sequence $\{\lambda_j\}_{j=0}^{\infty}$ such that the solution of (1.3) has exactly j zeros in $(0, \infty)$ and satisfies $\lim_{r\to\infty} r^{n-2}|u| < \infty$.*

We note here that the condition $rK(r) \in L^1(0, \infty)$ cannot be weakened in Theorem A. Indeed, if $rK(r) \notin L^1(0, 1)$, then (1.3) does not have a solution. Also, if $rK(r) \notin L^1(1, \infty)$, then any solution of (1.3) has infinitely many zeros in $(1, \infty)$.

In order to investigate nonradial eigenfunctions for (1.1), let us put $u(x) = v(r)\psi(z)$ with $r = |x|$ and $z \in S^{n-1}$. Substituting this in (1.1), we have

$$\Delta u + \lambda K(|x|)u = \left(v_{rr} + \frac{n-1}{r} v_r\right)\psi + \frac{v}{r^2}\Delta_z \psi + \lambda K(r) v\psi = 0,$$

where Δ_z is the Laplace-Beltrami operator on S^{n-1}. Hence

$$\frac{r^2}{v}\left(v_{rr} + \frac{n-1}{r} v_r\right) + \lambda r^2 K(r) = -\frac{\Delta_z \psi}{\psi} = \sigma$$

for some number σ. Thus we are led to the following two eigenvalue problems:

$$\Delta_z \psi + \sigma \psi = 0 \quad \text{in } S^{n-1} \tag{1.4}$$

and

$$v_{rr} + \frac{n-1}{r} v_r - \frac{\sigma}{r^2} v + \lambda K(r) v = 0. \quad r > 0, \tag{1.5}$$

It is known (see, e.g., Shimakura [10]) that all the eigenvalues of (1.4) are expressed as

$$\sigma_\ell = \ell(n-2+\ell), \quad \ell = 0, 1, 2, \cdots, \tag{1.6}$$

and the multiplicity p_ℓ of σ_ℓ is given by

$$p_\ell = (n-2+2\ell)(n-3+\ell)!/\{(n-2)!\ell!\}.$$

We denote by $\psi_\ell^{(m)}(z)$, $m = 1, 2, \cdots, p_\ell$, the normalized eigenfunctions associated with σ_ℓ which are orthogonal to each other. We note that the set $\{\{\psi_\ell^{(m)}(z)\}_{m=1}^{p_\ell}\}_{\ell=0}^{\infty}$ forms a complete orthonormal basis in $L^2(S^{n-1})$.

On the other hand, since $u(x) = v(r)\phi(z) \in \mathcal{D}$, we must find a solution of (1.5) satisfying

$$\int_{\mathbf{R}^n} r^{n-1} K(r) v^2 \, dr < \infty.$$

The term $-\sigma v/r^2$ in (1.5) does not allow a solution with $v(0) > 0$ unless $\sigma = 0$. So we seek a solution satisfying $v(t) = r^\beta + o(r^\beta)$ at $r = 0$ with suitable β. In the next section, we will show in Lemma 2.1 that (1.5) has a unique solution if and only if $\beta = \ell$.

Now we give our main results of this paper.

Theorem 1.1 *There exists a double sequence $\{\lambda_{k,\ell}\}_{k,\ell=0}^{\infty}$ such that (1.1) with $\lambda = \lambda_{k,\ell}$ has a solution of the form $v_{k,\ell}(r)\psi_\ell^{(m)}(z)$, where $v_{k,\ell}(r)$ is a solution of (1.5) with k zeros in $(0, \infty)$ such that $v_{k,\ell}(r) = r^\ell + o(r^\ell)$ at $r = 0$ and $\lim_{r\to\infty} r^{n+\ell-2}|v_{k,\ell}| < \infty$. Moreover the set $\{v_{k,\ell}(r)\{\psi_\ell^{(m)}(z)\}_{m=1}^{p_\ell}\}_{k,\ell=0}^{\infty}$ forms a complete orthogonal basis in $\mathcal{D}$.*

Concerning an ordering of the eigenvalues, we have the following result.

Theorem 1.2 *The eigenvalues of (1.1) satisfy*

$$0 < \lambda_{k,0} < \lambda_{k,1} < \lambda_{k,2} < \cdots \to \infty$$

for each $k \geq 0$, and

$$0 < \lambda_{0,\ell} < \lambda_{1,\ell} < \lambda_{2,\ell} < \cdots \to \infty$$

for each $\ell \geq 0$.

Remark. Here we only treat the case $rK(r) \in L^1(0,\infty)$, however, we can say more if $r^{n+2\ell-1}K(r) \in L^1(0,\infty)$ for some ℓ.

In this case, we can find another complete orthogonal basis. We will not give a proof of the following theorem here (see Kabeya and Yanagida [7]).

Theorem 1.3 *In addition to (K), suppose that $r^{n-1}K(r) \in L^1(0,\infty)$. Then there exists a complete orthogonal basis $\{v^{\phi}_{k,\ell}(r)\{\psi^{(m)}_{\ell}(z)\}_{m=1}^{p_\ell}\}_{k,\ell=0}^{\infty}$ which is uniquely determined by the following properties:*

(i) $v^{\phi}_{k,\ell}(r)$ *is a solution to (1.5) with* $\sigma = \sigma_\ell$ *and some* $\lambda = \lambda^{\phi}_k$.

(ii) $v^{\phi}_{k,\ell}(r)$ *satisfies* $v^{\phi}_{k,\ell}(r) = r^\ell + o(r^\ell)$ *at* $r = 0$ *and has exactly* k *zeros in* $(0,\infty)$.

(iii) *If* $r^{n+2\ell-1}K(r) \in L^1(0,\infty)$, *then* $w^{\phi}_{k,\ell}(r) := r^{-\ell}v^{\phi}_{k,\ell}(r)$ *satisfies*

$$\lim_{r\to\infty} \frac{-r^{n+2\ell-1}(w_{k,\ell})_r}{w_{k,\ell}} = \tan\phi_\ell, \quad k = 0,1,2,\cdots,$$

where $\phi_\ell \in [0,\pi/2]$ *is an arbitrarily given constant* ($\phi_\ell = \pi/2$ *means* $\lim_{r\to\infty} w^{\phi}_{k,\ell}(r) = 0$).

(iv) *If* $r^{n+2\ell-1}K(r) \notin L^1(0,\infty)$, *then* $v^{\phi}_{k,\ell}(r)$ *satisfies* $\lim_{r\to\infty} r^{n+\ell-2}|v^{\phi}_{k,\ell}| < \infty$.

2 Initial Value Problems

In this section, we investigate the equation

$$\begin{cases} v_{rr} + \dfrac{n-1}{r}v_r - \dfrac{\sigma_\ell}{r^2}v + \lambda K(r)v = 0, & r > 0 \\ v(r) = r^\beta + o(r^\beta) & \text{at } r = 0, \end{cases} \tag{2.1}$$

with some $\beta \geq 0$. First, we will choose a suitable β such that (2.1) has a unique solution.

Lemma 2.1 *The problem (2.1) has a unique solution if and only if* $\beta = \ell$ ($\ell \in \mathbf{N} \cup \{0\}$).

Proof. Put $v(r) = r^\beta w(r)$. Then we have

$$\frac{1}{r^{n-1}}\left\{r^{n-1}(\beta r^{\beta-1}w + r^\beta w_r)\right\}_r - \sigma_\ell r^{\beta-2}w + \lambda K(r) r^\beta w = 0.$$

Hence w satisfies

$$w_{rr} + \frac{n-1+2\beta}{r}w_r + \left\{\frac{\beta(\beta+n-2)-\sigma_\ell}{r^2} + \lambda K(r)\right\}w = 0, \quad (2.2)$$

$$w(0) = 1. \quad (2.3)$$

If $\beta(\beta+n-2)-\sigma_\ell \neq 0$, then

$$r\left\{\frac{\beta(n+\beta-2)-\sigma_\ell}{r^2} + \lambda K(r)\right\} \notin L^1(0,1).$$

In this case, any solution of (2.2) has infinitely many zeros as $r \downarrow 0$. Conversely, if $\beta(\beta+n-2)-\sigma_\ell = 0$, it is easy to show by using $rK(r) \in L^1(0,1)$ that (2.2)-(2.3) is solvable. By (1.6), the condition is rewritten as

$$\beta(\beta+n-2) - \ell(n-2+\ell) = (\beta-\ell)(\beta+n-2+\ell) = 0.$$

Thus $\beta = \ell$ must hold, because we seek bounded solutions near $r = 0$. $\square$

Lemma 2.2 *For each $\ell \geq 0$ and $k \geq 0$, there exists $\lambda_{k,\ell} > 0$ such that the unique solution $w(r;\lambda_{k,\ell})$ to (2.2)-(2.3) has exactly k zeros in $(0,\infty)$ with $\lim_{r\to\infty} r^{n+2\ell-2}|w| < \infty$, i.e., $r^\ell w(r;\lambda_{k,\ell}) \in \mathcal{D}$. Moreover, the inequalities*

$$\lambda_{0,\ell} < \lambda_{1,\ell} < \lambda_{2,\ell} < \cdots \to \infty$$

hold for each ℓ.

Proof. Since $\ell(\ell+n-2) - \sigma_\ell = 0$, we rewrite (2.2) as

$$(r^{n-1+2\ell}w_r)_r + \lambda r^{n-1+2\ell}K(r)w = 0. \quad (2.4)$$

Since $n-1+2\ell > 1$, we can apply Theorem A of Kabeya to show the existence of an increasing sequence $\{\lambda_k^{(\ell)}\}$ such that $w(r;\lambda_k^{(\ell)})$ has exactly k zeros in $(0,\infty)$ with $\lim_{r\to\infty} r^{n+2\ell-2}|w(r;\lambda_k^{(\ell)})| < \infty$ for $k \in \mathbf{N}\cup\{0\}$. $\square$

3 Existence of a Complete Basis

In this section, when $\ell \in \mathbf{N} \cup \{0\}$ is arbitrarily fixed, we prove the completeness of $\{w(r; \lambda_{k,\ell})\}$ in the class of radial functions. For simplicity of notation, let $\mathcal{D}_r^\alpha$ be a space of radial functions defined by

$$\mathcal{D}_r^\alpha = \{u \in C([0,\infty)) \mid \limsup_{r\to\infty} r^\alpha |u| < \infty\}.$$

Proposition 3.1 *Let $\ell \in \mathbf{N} \cup \{0\}$ be arbitrarily fixed. If a function $\varphi \in \mathcal{D}_r^{n+\ell-2}$ satisfies*

$$\int_0^\infty r^{n+2\ell-1} K(r)\varphi(r) w_\ell(r; \lambda_{k,\ell})\, dr = 0$$

for all $k \in \mathbf{N} \cup \{0\}$, then $\varphi \equiv 0$.

We will prove Proposition 3.1 by contradiction. To do so, we need sereval preliminary lemmas. First, we take a pair of fundamental solutions U and V to (2.4) which satisfy certain asymptotic behaviors at $r = 0$.

Let U be a solution to (2.4) with $U(0) = 1$. It is easy to see that such a solution exists for any $\lambda \in \mathbf{R}$. Then we define V so that V is a solution to (2.4) with $\max_{[1,\infty)} |V(r)| = 1$ and $V_r(1)/V(1) = U(1)/U_r(1)$. We agree that $V(1) = 0$ if $U_r(1) = 0$.

It is easy to see that the Wronskian $W(r) := U(r)V_r(r) - U_r(r)V(r) \neq 0$ at $r = 1$. Moreover, W satisfies

$$\frac{dW}{dr} + \frac{n + 2\ell - 1}{r} W = 0.$$

Hence W is given by

$$W(r) \equiv W(1) r^{n+2\ell-1} \neq 0 \quad \text{on } (0,\infty).$$

Thus, U and V are linearly independent of each other. This implies that V is singular at $r = 0$ and that $\lim_{r\downarrow 0} r^{n+2\ell-1} V_r \neq 0$.

Lemma 3.1 *Let $\ell \in \mathbf{N} \cup \{0\}$ be arbitrarily fixed. If a function $\varphi(r) \in \mathcal{D}_r^{n+\ell-2}$ satisfies*

$$\int_0^\infty r^{n+2\ell-1} K(r)\varphi(r) w_\ell(r; \lambda_{k,\ell})\, dr = 0$$

for all $k \in \mathbf{N} \cup \{0\}$, *then*

$$(r^{n+2\ell-1}u_r)_r + \lambda r^{n+2\ell-1}K(r)u = r^{n+2\ell-1}K(r)\varphi(r) \tag{3.1}$$

with

$$\lim_{r \downarrow 0} r^{n+2\ell-1}u_r = 0, \quad \lim_{r \to \infty} r^{n+\ell-3}|u| = 0 \tag{3.2}$$

has a solution $u(r;\lambda) \in C^2((0,\infty)) \cap C([0,\infty))$ *continuous with respect to* λ *for any* $\lambda \in \mathbf{R}$.

Proof. We follow the idea of the proof of Theorem 1 in §42 of Yosida [11]. Auxiliarily, we utilize two solutions linearly independent of each other to

$$(r^{n+2\ell-1}u_r)_r + \lambda r^{n+2\ell-1}K(r)u = 0. \tag{3.3}$$

Let $U(r;\lambda)$ be a solution to (3.3) with $U(0;\lambda) = 1$ and V be that as above. Let

$$\begin{aligned} w(r;\lambda) = -U \int_0^r W(s;\lambda)^{-1}K(s)\varphi(s)V(s)\,ds \\ +V \int_0^r W(s;\lambda)^{-1}K(s)\varphi(s)U(s)\,ds \end{aligned} \tag{3.4}$$

with

$$W(s;\lambda) = U(r;\lambda)V_r(r;\lambda) - U_r(r;\lambda)V(r;\lambda).$$

Since W satisfies

$$\frac{dW}{dr} + \frac{n+2\ell-1}{r}W = 0,$$

$W(r)$ is given by

$$W(r) \equiv c_\lambda r^{n+2\ell-1} \quad \text{on } (0,\infty)$$

with some $c_\lambda \in \mathbf{R}$ continuous with repsect to λ. Thus $w(r;\lambda)$ satisfies

$$\begin{aligned} w(r;\lambda) = c_\lambda^{-1}\Big\{ -U \int_0^r s^{n+2\ell-1}K(s)\varphi(s)V(s)\,ds \\ +V \int_0^r s^{n+2\ell-1}K(s)\varphi(s)U(s)\,ds\Big\}. \end{aligned} \tag{3.5}$$

Now we see that w is a solution to (3.1). Indeed, from direct calculation, we have

$$w_r = -U_r \int_0^r W^{-1}K\varphi V\,ds + V_r \int_0^r W^{-1}K\varphi U\,ds$$

and

$$(r^{n+2\ell-1}w_r)_r = -(r^{n+2\ell-1}U_r)_r \int_0^r W^{-1}K\varphi V\,ds$$
$$+(r^{n+2\ell-1}V_r)_r \int_0^r W^{-1}K\varphi U\,ds$$
$$+r^{n+2\ell-1}W^{-1}(UV_r - U_rV)K\varphi.$$

Thus we have

$$\begin{aligned}
&(r^{n+2\ell-1}w_r)_r + \lambda r^{n+2\ell-1}Kw \\
&\quad = \lambda r^{n+2\ell-1}KU \int_0^r W^{-1}K\varphi V\,ds \\
&\quad\quad -\lambda r^{n+2\ell-1}KV \int_0^r W^{-1}K\varphi U\,ds \\
&\quad\quad +\lambda r^{n+2\ell-1}K\left\{-U\int_0^r W^{-1}K\varphi V\,ds + V\int_0^r W^{-1}K\varphi U\,ds\right\} \\
&\quad\quad +r^{n+2\ell-1}K\varphi = r^{n+2\ell-1}K\varphi.
\end{aligned}$$

Thus any solution to (3.1) is expressed as

$$u(r;\lambda) = w(r;\lambda) + C_1(\lambda)U(r;\lambda) + C_2(\lambda)V(r;\lambda) \tag{3.6}$$

Since φ is bounded near $r=0$ and $rK(r) \in L^1(0,\infty)$, we have $s^{n+2\ell-1}K\varphi V$ and $s^{n+2\ell-1}K\varphi U \in L^1(0,1)$. Hence we have

$$\begin{aligned}
\lim_{r\downarrow 0} r^{n+2\ell-1}w_r &= c_\lambda^{-1}\lim_{r\downarrow 0}\left\{-r^{n+2\ell-1}U\int_0^r s^{n+2\ell-1}K(s)\varphi(s)V(s)\,ds\right. \\
&\qquad\qquad \left. +r^{n+2\ell-1}V\int_0^r s^{n+2\ell-1}K(s)\varphi\dot{H}(s)\,ds\right\} \\
&= 0.
\end{aligned}$$

Thus we have $C_2(\lambda) \equiv 0$ for any $\lambda \in \mathbf{R}$ since $\lim_{r\downarrow 0} r^{n+2\ell-1}V_r \neq 0$.

If $\lambda \neq \lambda_{k,\ell}$, then we have $\lim_{r\to\infty}|U(r;\lambda)| > 0$. Using the fact that $UV_r - U_rV = c_\lambda r^{-(n+2\ell-1)}$, we can show that $w \in L^\infty(0,\infty)$. Defining

$$C_1(\lambda) = c_\lambda^{-1}\int_0^\infty s^{n+2\ell-1}K\varphi V\,ds,$$

we get

$$\lim_{r\to\infty} u = \lim_{r\to\infty}(w + C_1(\lambda)U) = 0.$$

This implies that $C_1(\lambda)$ is also expressed as

$$C_1(\lambda) = -\lim_{r\to\infty} \frac{w(r;\lambda)}{U(r;\lambda)}.$$

Moreover, we have

$$\lim_{r\to\infty} r^{n+\ell-3}u = 0.$$

In case of $\lambda = \lambda_{k,\ell}$, $U^{(k)}(r) := U(r;\lambda_{k,\ell})$ is an eigenfunction and V satisfies $\lim_{r\to\infty}|V| > 0$. If otherwise, V must satisfy $\lim_{r\to\infty} r^{n+2\ell-2}|V| < \infty$. Then we come to a contradiction by applying the Kelvin transformation to (3.3) because (3.3) has a unique solution for each λ.

For a solution $u(r;\lambda)$ to (3.1) with $\lambda \neq \lambda_{k,\ell}$, we have

$$\int_0^\infty \left\{U^{(k)}(r^{n+2\ell-1}u_r)_r - u(r^{n+2\ell-1}U^{(k)}_r)_r\right\} dr = \left[r^{n+2\ell-1}(U^{(k)}u_r - u_r U^{(k)})\right]_0^\infty = 0$$

because $U^{(k)} \sim r^{2-(n+2\ell)}$, $U^{(k)}_r \sim r^{1-(n+2\ell)}$, $u = O(r^{2-(n+\ell)})$ and because $u_r = O(r^{1-(n+\ell)})$ at $r = \infty$. At $\lambda = \lambda_{k,\ell}$, since $\lim_{r\to\infty} V \neq 0$ and since $\lim_{r\to\infty} r^{n+2\ell-2}|U| < \infty$, we have

$$\lim_{r\to\infty} U^{(k)} \int_0^r s^{n+2\ell-1}K(s)\varphi V\, ds = 0$$

and

$$\lim_{r\to\infty} V \int_0^r s^{n+2\ell-1}K(s)\varphi U\, ds = 0$$

by assumption. Thus we obtain $\lim_{r\to\infty} w = 0$.

Moreover, we have $\lim_{r\to\infty} r^{n+\ell-3}u = 0$ irrelevent to $C_1(\lambda)$.

To determine $C_1(\lambda)$, we need another expresion of $C_1(\lambda)$. From (3.1) and (3.3), we get

$$\begin{aligned} 0 &= \int_0^\infty \left\{U^{(k)}(r^{n+2\ell-1}u_r)_r - u(r^{n+2\ell-1}U^{(k)}_r)_r\right\} dr \\ &= (\lambda_{k,\ell} - \lambda) \int_0^\infty r^{n+2\ell-1}KuU^{(k)}\, dr + \int_0^\infty r^{n+2\ell-1}KU^{(k)}\varphi\, dr \\ &= (\lambda_{k,\ell} - \lambda) \int_0^\infty r^{n+2\ell-1}KuU^{(k)}\, dr \end{aligned}$$

by assumption. Hence we have

$$\int_0^\infty r^{n+2\ell-1}KuU^{(k)}\, dr = 0. \tag{3.7}$$

Substituting $u = w(r;\lambda) + C_1(\lambda)U(r;\lambda)$ for (3.7), we obtain

$$C_1(\lambda) = -\frac{\displaystyle\int_0^\infty r^{n+2\ell-1}Kw(r;\lambda)U^{(k)}\,dr}{\displaystyle\int_0^\infty r^{n+2\ell-1}KU(r;\lambda)U^{(k)}\,dr}.$$

Although we should be careful when letting $\lambda \to \lambda_{k,\ell}$, we manage to have

$$\lim_{\lambda\to\lambda_{k,\ell}} C_1(\lambda) = -\frac{\displaystyle\int_0^\infty r^{n+2\ell-1}Kw(r;\lambda_{k,\ell})U^{(k)}\,dr}{\displaystyle\int_0^\infty r^{n+2\ell-1}K(U^{(k)})^2\,dr}.$$

The right-hand side is the desired form of $C_1(\lambda_{k,\ell})$. This shows the continuity of $u(r;\lambda)$ with respect to λ. □

Lemma 3.2 *The problem*

$$\begin{cases} (r^{n+2\ell-1}u_r)_r + \lambda r^{n+2\ell-1}K(r)u = r^{n+2\ell-1}K(r)\varphi \\ \\ \lim\limits_{r\downarrow 0} r^{n+2\ell-1}u_r(r) = 0, \quad \lim\limits_{r\to\infty} r^{n+\ell-3}|u(r)| = 0. \end{cases} \tag{3.8}$$

with $\varphi \not\equiv 0$ cannot have a unique solution $u(r;\lambda)$ for all $\lambda \in \mathbf{R}$.

Proof. Suppose to the contrary that (3.8) has a unique solution for any $\lambda \in \mathbf{R}$. We will show that the the linearized equation

$$(r^{n+2\ell-1}\hat{w}_r)_r + r^{n+2\ell-1}K(r)u + \lambda r^{n+2\ell-1}K(r)\hat{w} = 0 \tag{3.9}$$

with

$$\lim_{r\to\infty} r^{n+\ell-3}\hat{w}_r(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} r^{n+\ell-3}\hat{w} = 0$$

has only a trivial solution $\hat{w} \equiv 0$. Hence u must satisfy $u \equiv 0$ and from (3.8), we obtain $\varphi \equiv 0$, which is a contradiction.

In case of $\lambda = 0$, $u(r;0)$ satisfies

$$\begin{cases} (r^{n+2\ell-1}u_r(r;0))_r = r^{n+2\ell-1}K(r)\varphi, \\ \\ \lim\limits_{r\to\infty} r^{n+2\ell-1}u_r(r;0) = 0, \quad \limsup\limits_{r\to\infty} r^{n+\ell-2}|u(r;0)| < \infty. \end{cases} \tag{3.10}$$

Hence we have

$$\int_0^\infty \left\{u(r;0)(r^{n+2\ell-1}\hat{w}_r)_r - \hat{w}(r^{n+2\ell-1}u_r(r;0))_r\right\} dr$$
$$= -\lambda \int_0^\infty r^{n+2\ell-1}K\hat{w}u(r;0)\,dr - \int_0^\infty r^{n+2\ell-1}Ku(r;\lambda)u(r;0)\,dr$$
$$- \int_0^\infty r^{n+2\ell-1}K\varphi\hat{w}\,dr.$$

The left-hand side yields

$$\begin{aligned}\int_0^\infty &\left\{u(r;0)(r^{n+2\ell-1}\hat{w}_r)_r - \hat{w}(r^{n+2\ell-1}u_r(r;0))_r\right\} dr \\ &= \left[u(r;0)r^{n+2\ell-1}\hat{w}_r - \hat{w}r^{n+2\ell-1}u_r(r;0)\right]_0^\infty \\ &= 0\end{aligned} \tag{3.11}$$

by (3.10) and $\lim_{r\to\infty} r^{\ell+1}\hat{w}_r = 0$ (since $\lim_{r\to\infty} r^{n+\ell-2}\hat{w}_r = 0$ and $n \geq 3$). Thus we obtain

$$\begin{aligned}\lambda \int_0^\infty r^{n+2\ell-1}K\hat{w}u(r;0)\,dr &+ \int_0^\infty r^{n+2\ell-1}Ku(r;\lambda)u(r;0)\,dr \\ &= -\int_0^\infty r^{n+2\ell-1}K\varphi\hat{w}\,dr.\end{aligned} \tag{3.12}$$

Similarly, we have

$$\begin{aligned}\int_0^\infty &\left\{u(r;\lambda)(r^{n+2\ell-1}\hat{w}_r)_r - \hat{w}(r^{n+2\ell-1}u_r(r;\lambda))_r\right\} dr \\ &= \left[u(r;\lambda)r^{n+2\ell-1}\hat{w}_r - \hat{w}r^{n+2\ell-1}u_r(r;\lambda)\right]_0^\infty = 0.\end{aligned} \tag{3.13}$$

From (3.8) and (3.9), the left-hand side yields

$$\int_0^\infty \left\{u(r;\lambda)(r^{n+2\ell-1}\hat{w}_r)_r - \hat{w}(r^{n+2\ell-1}u_r(r;\lambda))_r\right\} dr$$
$$= -\int_0^\infty r^{n+2\ell-1}Ku(r;\lambda)^2\,dr - \int_0^\infty r^{n+2\ell-1}K\varphi\hat{w}\,dr.$$

Thus we get

$$\int_0^\infty r^{n+2\ell-1}K(r)u(r;\lambda)^2\,dr = -\int_0^\infty r^{n+2\ell-1}K\varphi\hat{w}\,dr. \tag{3.14}$$

Let

$$v(\lambda) := \lambda \int_0^\infty r^{n+2\ell-1} K u(r;0) u(r;\lambda)\, dr. \tag{3.15}$$

Then

$$v'(\lambda) = \int_0^\infty r^{n+2\ell-1} K u(r;0) u(r;\lambda)\, dr + \lambda \int_0^\infty r^{n+2\ell-1} K u(r;0) \hat{w}\, dr.$$

By (3.12), we have

$$v'(\lambda) = -\int_0^\infty r^{n+2\ell-1} K \varphi \hat{w}\, dr.$$

Combining this with (3.14), we obtain

$$v'(\lambda) = \int_0^\infty r^{n+2\ell-1} K u(r;\lambda)^2\, dr \geq 0.$$

If $v'(\lambda) = 0$, then we have $u(r;\lambda) \equiv 0$. Thus we get $\hat{w} \equiv 0$, the desired assertion. So we consider the case $v'(\lambda) > 0$. From the definition of $v(\lambda)$, $v(0) = 0$ and hence we have

$$v(\lambda) > 0 \quad \text{for } \lambda > 0,$$

$$v(\lambda) < 0 \quad \text{for } \lambda < 0.$$

Applying the Schwarz inequality to (3.15),

$$v(\lambda)^2 \leq c_0^2 \lambda^2 v'(\lambda) \tag{3.16}$$

with $c_0^2 = \int_0^\infty r^{n+2\ell-1} K(r) u(r;0)^2\, dr$. The inequality (3.16) implies

$$\frac{d}{d\lambda}\left\{\frac{1}{\lambda} - \frac{c_0^2}{v(\lambda)}\right\} = -\frac{1}{\lambda^2}c_0^2 + \frac{v'(\lambda)}{v(\lambda)^2} \geq 0 \tag{3.17}$$

Since $v(\lambda)$ is monotone increasing in λ, we have

$$\frac{1}{\lambda} - \frac{c_0^2}{v(\lambda)} \leq \lim_{r\to\infty}\left\{\frac{1}{\lambda} - \frac{c_0^2}{v(\lambda)}\right\} \leq 0, \tag{3.18}$$

i.e.,

$$v(\lambda) \leq c_0^2 \lambda \quad \text{for } \lambda \geq 0. \tag{3.19}$$

Similarly, for $\lambda < 0$, we have

$$v(\lambda) \geq c_0^2 \lambda \quad \text{for } \lambda < 0. \tag{3.20}$$

We also get

$$\lim_{\lambda \to 0} \left\{ \frac{1}{\lambda} - \frac{c_0^2}{v(\lambda)} \right\} = 0$$

by (3.17). With some consideration, we obtain

$$v(\lambda) \equiv c_0^2 \lambda \quad \text{for } \lambda \in \mathbf{R}. \tag{3.21}$$

From (3.21), we have

$$v'(\lambda) = c_0^2 = v'(0) = \frac{v(\lambda)}{\lambda}.$$

In view of the definitions of $v(\lambda)$ and c_0^2, we get

$$\begin{aligned} \int_0^\infty s^{n+2\ell-1} K u(s;0) u(s;\lambda)\, ds &= \int_0^\infty s^{n+2\ell-1} K u(s;\lambda)^2\, ds \\ &= \int_0^\infty s^{n+2\ell-1} K u(s;0)^2\, ds. \end{aligned}$$

Thus we obtain

$$\int_0^\infty s^{n+2\ell-1} K (u(s;0) - u(s;\lambda))^2\, ds = 0,$$

which implies that $u(r;0) \equiv u(r;\lambda)$, i.e., $\hat{w} \equiv 0$, a contradiction. □

Proof of Proposition 3.1. The statement of Lemma 3.2 cantradicts that of Lemma 3.1. Thus we see that $\varphi \equiv 0$. □

4 Proof of Theorems

To prove Theorem 1.1, we need to show that

$$\mathcal{D} = \{u \mid u \text{ is measurable}, \int_{\mathbf{R}^n} K(|x|) u^2\, dx < \infty\}$$

is a separable Hilbert space and that $\{v_{k.\ell}(r)\{\psi_\ell^{(m)}(z)\}_{m=1}^{p_\ell}\}_{k,\ell=0}^\infty$ forms an orthogonal basis, where $v_{k,l} := r^\ell w_k(r;\lambda_{k,\ell})$ with $w_k(r;\lambda_{k,\ell})$ defined just below the proof of Lemma 2.1 and $\psi_\ell^{(m)}(z)$ is an eigenfunction of $-\Lambda$ corresponding to the eigenvalue $\sigma_\ell = \ell(n-2+\ell)$ with $1 \le m \le p_\ell = (n-2+2\ell)(n-3+\ell)!/\{(n-2)!\ell!\}$.

Proposition 4.1 *Under* (K), *$\mathcal{D}$ is a separable Hilbert space with its inner product*

$$(u,v) = \int_{\mathbf{R}^n} K(|x|)uv\,dx \quad (u,v \in \mathcal{D}).$$

Moreover, $\mathcal{D}_r \otimes L^2(S^{n-1})$ is dense in $\mathcal{D}$ and $\{v_{k.\ell}(r)\{\psi_\ell^{(m)}(z)\}_{m=1}^{p_\ell}\}_{k,\ell=0}^\infty$ is a complete orthogonal basis to $\mathcal{D}$.

Proof. It is easy to see that $\mathcal{D}$ is a separable Hilbert space and $\mathcal{D}_r \otimes L^2(S^{n-1})$ is dense in $\mathcal{D}$. Since it is well-known that $L^2(S^{n-1})$ has a complete countable orthogonal basis $\{\{\psi_\ell^{(m)}(z)\}_{m=1}^{p_\ell}\}^\infty$ (see,e.g. Shimakura [10]), we have only to show the ortho-normality of $\{v_{k,\ell}\}$.

As for $\mathcal{D}_r$, it is easy to see that

$$\int_0^\infty r^{n-1}K(r)v_{k,\ell}v_{j,\ell}\,dr = 0 \tag{4.1}$$

for any $\ell = 0,1,2,\dots$ and $k \ne j$. Indeed, since $v_{k,\ell} \in \mathcal{D}_r$ with $v_{k,\ell}(r) \sim r^\ell$ at $r=0$ is an eigenfunction for the eigenvalue $\lambda_{k,\ell}$, we have

$$\begin{aligned}
\int_0^\infty r^{n-1}K(r)v_{k,\ell}v_{j,\ell}\,dr &= -\frac{1}{\lambda_{k,\ell}}\int_0^\infty (r^{n-1}v'_{k,\ell})'v_{j,\ell}\,dr \\
&= -\frac{1}{\lambda_{k,\ell}}\int_0^\infty v_{k,\ell}(r^{n-1}v'_{j,\ell})'\,dr \\
&= \frac{\lambda_j^{(\ell)}}{\lambda_{k,\ell}}\int_0^\infty r^{n-1}K(r)v_{k,\ell}v_{j,\ell}\,dr.
\end{aligned}$$

Since $\lambda_{k,\ell} \ne \lambda_j^{(\ell)}$, (4.1) is proved. From Proposition 3.1, we see that $\{v_{k,\ell}\}$ forms a complete basis. Thus we see that $\{v_{k,\ell}(r)\{\psi_\ell^{(m)}(z)\}_{m=1}^{p_\ell}\}_{k=0,\ell=0}^\infty$ is a countable dense set in $\mathcal{D}$. The proof is complete □

Proof of Theorem 1.1 If $\ell \ne \ell'$, then the orthogonality and the completeness comes from those of $\{\psi_\ell^{(m)}\}$. If $\ell = \ell'$, then the conclusion is a direct consequence of Proposition 4.1. □

Proof of Theorem 1.2 The relation $\lambda_{0,\ell} < \lambda_{1,\ell} < \lambda_{2,\ell} < \ldots < \lambda_{k,\ell} < \lambda_{k+1,\ell} < \ldots$ comes from Lemma 2.2 and $\lambda_{k,0} < \lambda_{k,1} < \lambda_{k,2} < \ldots < \lambda_{k,\ell} < \lambda_{k,\ell+1} < \ldots$ is an easy consequence of Sturm's comparison Theorem. □

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